Incentives for the Over-Provision of Public Goods: Supplemental Appendix for Online Publication

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A Illustrations Using a Closed-Form Utility Function

In this online appendix, I highlight the main results by applying a simple functional form to the utility function that offers closed-form solutions. The utility when contributing is given by

$$u(x_i, X; \sigma) = X - \gamma \frac{1}{40} X^2 + \sigma \left(\lambda x_i + (1 - \lambda) \frac{x_i}{X} \right) - \frac{1}{2} x_i^2. \tag{A.1}$$

Free-riders receive utility

$$u(0, X; \sigma) = \max \left\{ X - \gamma \frac{1}{40} X^2, 0 \right\}.$$

The remainder of the online appendix is subdivided by the type of public good followed by the extension, as in the main paper.

A.1 Public Goods with Costly Consumption ($\gamma = 1$ and $\sigma = 0$)

I first illustrate Proposition 1. Suppose that $\gamma = 1$ and $\sigma = 0$. Then by (A.1), each contributor's utility is

$$u(x_i, X; 0) = X - \frac{1}{40} X^2 - \frac{1}{2} x_i^2.$$

It follows that

$$X^* = N \Bigg(\underbrace{\frac{20}{20 + N}}_{x^*} \Bigg)$$

$$\tilde{X} = N \Bigg(\underbrace{\frac{20N}{20 + N^2}}_{\tilde{x}} \Bigg)$$

$$\hat{X} = 20.$$

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Whenever $N > 1$, $X^* < \tilde{X}$. To illustrate the non-uniform convergence in the free-rider problem discussed in the text, note that

$$\tilde{X} - X^* = \frac{400(N-1)}{(20+N)(20+N^2)},$$

which is decreasing in N only for $N \geq 5$. Lastly, as N increases, both X^* and $\tilde{X}$ converge to $\hat{X} = 20$.

Next, note that while the free-rider problem is less severe when $\gamma = 1$, utility is still higher under $\gamma = 0$. Using (A.1), when $\gamma = 0$, $x^* = 1$ and $\tilde{x} = N$. Thus, the respective utilities in the equilibrium and welfare maximizing case are

$$\begin{aligned} u(x^*, X^*; \sigma) &= N - \frac{1}{2} \\ u(\tilde{x}, \tilde{X}; \sigma) &= \frac{N^2}{2}. \end{aligned}$$

When $\gamma = 1$, the respective utilities in the equilibrium and welfare maximizing case are

$$\begin{aligned} u(x^*, X^*; \sigma) &= 10 - \frac{4200}{(20+N)^2} \\ u(\tilde{x}, \tilde{X}; \sigma) &= \frac{10N^2}{20+N^2}. \end{aligned}$$

By inspection, each utility is higher under $\gamma = 0$.

A.2 Public Goods with Private Benefits ($\gamma = 0$ and $\sigma > 0$)

To illustrate Proposition 2(i) and 2(ii), suppose that $\gamma = 0$, $\sigma > 0$, and $\lambda = 1$. Then by (A.1), each contributor's utility is

$$u(x_i, X, \sigma) = X + \sigma x_i - \frac{1}{2}x_i^2.$$

Aggregate contributions are given by

$$\begin{aligned} X^* &= N(\underbrace{1+\sigma}_{x^*}) \\ \tilde{X} &= N(\underbrace{N+\sigma}_{\tilde{x}}). \end{aligned}$$

Contributions are increasing in σ and free-riding persists and worsens as the population size grows:

$$\tilde{X} - X^* = N(N-1).$$

Hence, under-provision occurs as predicted regardless of σ.

To illustrate Proposition 2(iii) and 2(iv), suppose that $\lambda = 0$, so each contributor's utility is

$$u(x_i, X; \sigma) = X + \sigma \frac{x_i}{X} - \frac{1}{2}x_i^2.$$

The Nash equilibrium and welfare-maximizing contributions are

$$X^* = N\Bigg(\underbrace{\frac{1}{2} + \sqrt{\frac{1}{4} + \sigma\frac{N-1}{N^2}}}_{x^*}\Bigg)$$
$$\tilde{X} = N \times \underbrace{N}_{\tilde{x}}.$$

As in Proposition 2(iii), $\tilde{X}$ is unaffected by σ while X^* is strictly increasing in σ. Eventually, X^* must exceed $\tilde{X}$. Equating X^* and $\tilde{X}$ and solving for σ yields the cutoff

$$\overline{\sigma}(N) = N^3,$$

which is increasing in N as expected given the linearity of $b\left(\frac{x_i}{X}\right)$.

I now illustrate Proposition 3 and how the cutoff values relate between Propositions 2 and Proposition 3. Suppose that the number of contributors bounded above by M'. Then,

$$X^* = M'\Bigg(\underbrace{\frac{1}{2} + \sqrt{\frac{1}{4} + \sigma\frac{M'-1}{(M')^2}}}_{x^*}\Bigg)$$
$$\tilde{X} = M' \times \underbrace{N}_{\tilde{x}}.$$

Equating these two values and solving for σ yields

$$\overline{\overline{\sigma}}(M', N) = \frac{(M')^2}{M'-1}N(N-1).$$

Note that,

$$\overline{\overline{\sigma}}(M', N) < \overline{\sigma}(N) \iff M' > \frac{N}{N-1},$$

which is true for $2 \leq M' < N$.

Note that the ordering of the cutoff values in Propositions 2 and 3 need not be true in general. That is, the minimum σ to generate over-provision when only a subset of the population can contribute need not be less than the minimum σ when the entire population can contribute. Specifically, the relationship depends on the shape of $b(1/M)$, particularly its slope as M increases.

A.3 Public Goods with Costly Consumption and Private Benefits ($\gamma = 1$ and $\sigma > 0$)

This section highlights the results of Proposition 4. Suppose that $\gamma = 1$ and $\sigma > 0$. Then by (A.1), each contributor's utility is

$$u(x_i, X; \sigma) = X - \frac{1}{40}X^2 + \sigma\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right) - \frac{1}{2}x^2,$$

leading to contributions of

$$X^* = \begin{cases} N\left(\frac{20(1+\sigma)}{20+N}\right) & \text{if } \lambda = 1 \\ N\left(\frac{2\left(5N+\sqrt{(5N)^2+5\sigma(N+20)(N-1)}\right)}{N(20+N)}\right) & \text{if } \lambda = 0, \end{cases}$$

$$\tilde{X} = \begin{cases} N\left(\frac{20(N+\sigma)}{20+N^2}\right) & \text{if } \lambda = 1 \\ N\left(\frac{20N}{20+N^2}\right) & \text{if } \lambda = 0. \end{cases}$$

The welfare maximizer always converges to $20 = \hat{X}$ as $N \to \infty$, while

$$\lim_{N\to\infty} X^* = \begin{cases} 20 + \sigma & \text{if } \lambda = 1 \\ 10 + 2\sqrt{5(5+\sigma)} & \text{if } \lambda = 0. \end{cases}$$

Both values are strictly greater than 20 for all $\sigma > 0$. Equating X^* and $\tilde{X}$ and solving for σ yields

$$\overline{\sigma}(N) = \begin{cases} \frac{20}{N} & \text{if } \lambda = 1 \\ \frac{20}{N}\left(\frac{2\sqrt{5}N^2}{20+N^2}\right)^2 & \text{if } \lambda = 0. \end{cases}$$

In both cases, $\lim_{N\to\infty} \overline{\sigma}(N) = 0$; however, when contribution benefits are derived from shares, $\overline{\sigma}(N)$ is increasing if $N \leq 7$ and decreasing if $N \geq 8$.

A.4 Extension: Hybrid Private Benefits to Contributors

This section utilizes the assumptions of the extension in Section 5 of the main text. First, suppose that $\gamma = 0$ so by (A.1), contributor utility is given by

$$u(x_i, X; \sigma) = X + \sigma\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right) - \frac{1}{2}x_i^2.$$

The equilibrium and welfare-maximizing contributions are given by

$$X^* = N\Bigg(\underbrace{\frac{1+\lambda\sigma}{2} + \sqrt{\frac{(1-\lambda\sigma)^2}{4} + \sigma\frac{(1-\lambda)(N-1)}{N^2}}}_{x^*}\Bigg)$$
$$\tilde{X} = N\,\underbrace{(N+\lambda\sigma)}_{\tilde{x}}.$$

To demonstrate over-provision, note that $X^* = \tilde{X}$ if and only if

$$\sigma = \frac{N^3}{1-\lambda(1+N^2)} \equiv \overline{\sigma}(N,\lambda),$$

which is well defined (as $\sigma \geq 0$) only if $\lambda < \frac{1}{1+N^2}$. Thus, over-provision persists so long as $\lambda < \frac{1}{1+N^2}$ and $\sigma > \frac{N^3}{1-\lambda(1+N^2)}$.

Now suppose that $\gamma = 1$, so by (A.1), contributor utility is

$$u(x_i, X; \sigma) = X - \frac{1}{40}X^2 + \sigma\left(\lambda x_i + (1-\lambda)\frac{x_i}{X}\right) - \frac{1}{2}x_i^2.$$

The equilibrium and welfare-maximizing contributions are given by

$$X^* = N\Bigg(\underbrace{\frac{10(1+\lambda\sigma)}{20+N} + 2\sqrt{5}\sqrt{\frac{(19N-20)(1-\lambda)\sigma}{N^2(20+N)} + \frac{5+\sigma+\lambda\sigma(9+5\lambda\sigma)}{20+N}}}_{x^*}\Bigg)$$
$$\tilde{X} = N\Bigg(\underbrace{\frac{20(N+\lambda\sigma)}{20+N^2}}_{\tilde{x}}\Bigg)$$

Equating these two values and solving for σ yields the cutoff

$$\overline{\sigma}(N,\lambda) = \frac{-400(1-\lambda)}{40\lambda^2N^3} - \frac{1}{40\lambda N^2}\Bigg[40 - 440\lambda + N\Bigg(N + 19\lambda N - \sqrt{\frac{(20-N^2)^2\left(400(1-\lambda)^2 + N^4(1+19\lambda)^2 - 40N^2(1-\lambda)(21\lambda-1)\right)}{N^6}}\Bigg)\Bigg].$$

As $\lambda \to 1$, $\overline{\sigma}(N,\lambda) \to \frac{20}{N^2}$, which corresponds to the cutoff in Section A.3.